COMMON FIXED POINT THEOREMS USING PROP. (E.A) IN COMPLEX VALUED b-METRIC SPACES

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ABSTRACT:-Azam et al. [3] introduced the notion of complex valued metric space and obtained a common fixed point result for mappings satisfying rational inequality in this space. Czerwik [9] proved a contraction theorem in b-metric space which generalized the Banach contraction principal. In this paper, we prove some common fixed point theorems for two pairs of weakly compatible mappings satisfying property (E.A) for a set of rational inequalities in complex valued b-metric space. Further, we discuss about the common fixed point theorem applying entirely within the closed ball instead of whole space (e.g., [2]). Our result thus generalizes many results existing in the literature, e.g. [2]-[5], [15] etc.

Key Words: - Banach contraction principal, common fixed point, complete metric space, complex valued metric space, complex valued b-metric space, weakly compatible mappings.

AMS Classification: 47H10, 54H25.

1. INTRODUCTION

Banach contraction principal [6] is a basic result in fixed point theory. This theorem has been generalized in many ways. Bakhtin [7] introduced the notion of b-metric space as a generalization of metric space in which the triangle inequality is relaxed. Further, Czerwik [9] proved a contraction theorem in this space which generalized the Banach contraction principal. Malhotra and Bansal [16] proved some common coupled fixed point theorems for generalized contractive mappings in b-metric spaces.

Azam et. al. [3] have introduced the complex valued metric space which is a generalization of the metric space. They obtained some fixed point results for a pair of mappings for a contraction condition satisfying a rational inequality which are not meaningful in cone metric spaces. Therefore many results of analysis can’t be generalized to cone metric space. Further, Bhatt et al. [4]-[5] generalized the result of Azam et al. [3]. On the other hand, Jungck [10] introduced compatible mappings for a pair of mappings. In this line, Pant [18] proved some fixed point results using noncompatible mappings. A further generalization of compatible mappings, namely weakly compatible mappings, was introduced by Jungck [11]. Many fixed point results have been proved using this notion. The non-compatibility was further relaxed to property (E.A) ([1]). Verma and Pathak [26] proved a fixed point result using property (E.A) in complex valued metric space. More results on complex valued metric spaces can be found in [2], [3], [4], [5], [8], [12], [13], [14], [17], [19], [21], [22], [23], [24], [25], [26] and [27] etc.

In 2014, Mukheimer [15] and Rao et al. [20] proved some common fixed point theorems using complex valued b-metric space. In this paper, we prove a common fixed point theorem for two pairs of weakly compatible mappings in a complete complex valued b-metric space.
satisfying a set of contraction condition. Our theorem generalizes many results existing in the literature.

2. PRELIMINARIES

Let $C$ is the set of complex numbers $z = a+ib$. Here $a, b$, are real numbers, $a$ is called Re$(z)$ and $b$ is called Im$(z)$. A complex valued metric $d$ is a function from a set $X$ into $C$. Let $z_1, z_2 \in C$; define a partial order $\leq$ on $C$ as follows:

$$z_1 \leq z_2 \text{ if and only if } \text{Re}(z_1) \leq \text{Re}(z_2), \text{Im}(z_1) \leq \text{Im}(z_2).$$

It follows that $z_1 \leq z_2$ if one of the following conditions is satisfy:

(i) $\text{Re}(z_1) = \text{Re}(z_2), \text{Im}(z_1) < \text{Im}(z_2),$
(ii) $\text{Re}(z_1) < \text{Re}(z_2), \text{Im}(z_1) = \text{Im}(z_2),$
(iii) $\text{Re}(z_1) < \text{Re}(z_2), \text{Im}(z_1) < \text{Im}(z_2),$
(iv) $\text{Re}(z_1) = \text{Re}(z_2), \text{Im}(z_1) = \text{Im}(z_2).$

In (i), (ii) and (iii), we have $|z_1| < |z_2|$. In (iv), we have $|z_1| = |z_2|$. So $|z_1| \leq |z_2|$, whenever $z_1 \leq z_2$. In particular, $z_1 \leq z_2$ if $z_1 \neq z_2$ and one of (i), (ii), (iii) satisfy. In this case $|z_1| < |z_2|$. We will write $z_1 \leq z_2$ if only (iii) satisfy. Hence $z_1 \leq z_2 \Rightarrow |z_1| < |z_2|.$

Remark. ([12]) We note that the following statements hold:

(i) $a, b \in R$ and $a \leq b \Rightarrow az \leq bz, \forall z \in C;
(ii) 0 \leq z_1 \neq z_2 \Rightarrow |z_1| < |z_2|, \forall z_1, z_2 \in C;
(iii) $z_1 \leq z_2$ and $z_2 \leq z_3 \Rightarrow z_1 \leq z_3, \forall z_1, z_2, z_3 \in C.$

Azam et al. [3] defined complex-valued metric space $(X, d)$ in the following way:

**Definition 2.1.** [3] Let $X$ be a nonempty set. Suppose that the mapping $d: X \times X \rightarrow C$ satisfy the following conditions:

(A1) $0 \leq d(x, y), d(x, y) = 0$ if and only if $x = y, \forall x, y \in X,$
(A2) $d(x, y) = d(y, x), \forall x, y \in X,$
(A3) $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X,$
then $d$ is called complex-valued metric, and $(X, d)$ is called a complex-valued metric space.

**Definition 2.2.** ([16]) Let $X$ be a nonempty set and let $s \geq 1$ be a real number. The mapping $d: X \times X \rightarrow [0, 1]$ is called b-metric space if:

(B1) $0 \leq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y,$
(B2) $d(x, y) = d(y, x),$
(B3) $d(x, y) \leq s[d(x, z) + d(z, y)].$
Satisfy; for all $x, y, z \in X.$ The number $s$ is called the coefficient of b-metric space.

**Example 2.3.** ([16]) Let $(X, d)$ be a metric space and $\rho(x, y) = (d(x, y))^p, p > 1.$ Then $(X, \rho)$ is a b-metric space with $s = 2p-1.$

On generalizing the above (A3), i.e., the triangle inequality, in the following way, Complex valued b-metric space is defined as follows:

**Definition 2.4.** ([15], [16], [20]) Let $X$ be a nonempty set. Suppose that the mapping $d: X \times X \rightarrow C$ satisfy the following conditions:

(C1) $0 \leq d(x, y), d(x, y) = 0,$ and only if $x = y,$
for all $x, y \in X,$
(C2) $d(x, y) = d(y, x), \forall x, y \in X,$
(C3) $d(x, y) \leq s[d(x, z) + d(z, y)], \forall x, y, z \in X,$
where $s \geq 1$ is a real number, then $d$ is called a complex valued b-metric on $X,$ and $(X, d)$ is called complex valued b-metric space.

Remark. Note that if $s = 1$ then the complex valued b-metric space reduces to a complex valued metric space. Thus every complex valued metric space is a complex valued b-metric space, but not conversely. This generalizes the concept of a complex valued b-metric space over complex valued metric space.

**Example 2.5.** ([20]) Let $X = [0, 1].$ Define a complex valued metric $d: X \times X \rightarrow C$ by:

$$d(x, y) = |x−y|^2+i|x−y|^2, \forall x, y \in X;$$
then $(X, d)$ is a complex valued b-metric space with $s=2.$

**Definition 2.6.** ([20]) Let $(X, d)$ be a complex valued b-metric space, then

(i) A point $x \in X$ is called interior point of a set $A \subseteq X$ whenever $\exists 0 < r \in C:$

$$B(x, r) = \{y \in X: d(x, y) < r\} \subseteq A.$$
(ii) A point \( x \in X \) is called the limit point of a set \( A \subseteq X \) whenever for every \( 0 < r \in \mathbb{C} : B(x, r) \cap (X-A) \neq \emptyset \).

(iii) A subset \( B \subseteq X \) is called open whenever each element of \( B \) is an interior point of \( B \).

(iv) A subset \( B \subseteq X \) is called closed whenever each limit point of \( B \) belongs to \( B \).

(v) The family \( F = \{ B(x, r) : x \in X, 0 < r < 1 \} \) is a sub-basis for a topology on \( X \).

We denote this complex topology by \( T_c \). Indeed the topology \( T_c \) is Hausdorff.

**Definition 2.7.** [11] A pair of self-mappings \( A, B: X \rightarrow X \) is called weakly compatible if they commute at their coincidence points. That is, if there be a point \( u \in X \) such that \( Au = Su \), then \( ASu = SAu \), for each \( u \in X \).

**Example 2.8.** Let \( X = \mathbb{R} \). Define a complex valued b-metric \( d:X \times X \rightarrow \mathbb{C} \) by:

\[
d(x, y) = |x - y|^2 + i|x - y|^2, \quad \forall \, x, y \in X. \]

Define mappings \( f, g:X \rightarrow X \) by: \( fx = x/4 \); \( gx = x/5 \), \( \forall x \in X \).

Then \( f \) and \( g \) have coincidence point at \( x = 0 \). Now at this point, \( fg0 = gf0 \). Thus \((f, g)\) is weakly compatible at 0.

Mukheimer [15] proved the following common fixed point theorem in complex-valued b-metric space to generalize the main theorem of Azam et al. [3]:

**Theorem 2.9.** ([15]) Let \((X, d)\) be a complete complex valued b-metric space with the coefficient \( s \geq 1 \) and let \( S, T : X \rightarrow X \) be mappings satisfying:

\[
d(Sx, Ty) \leq \alpha d(x, y) + \mu d(Sx, y) + d(Ty, Ty)/[1+d(x, y)]\]

\( \quad \forall x, y \in X, \) \( \alpha, \mu \) are non-negative real numbers with \( s \alpha + \mu < 1 \). Then \( S, T \) have a unique common fixed point in \( X \).

**Theorem 2.10.** ([15]) Let \((X, d)\) be a complete complex valued b-metric space with the coefficient \( s \geq 1 \) and let \( S, T : X \rightarrow X \) be mappings satisfying:

\[
d(Sx, Ty) \leq a [d(x, Sx)d(x, Ty) + d(y, Ty)d(y, Sx)]/[d(x, Ty) + d(y, Sx)]\]

\( \quad \forall x, y \in X \) \( \leq \) \( \alpha \lambda + \beta \). \( (2.2) \)

Ahmad et al. [2] proved the following result for a pair of mappings in a complete complex valued metric space satisfying a contractive condition on the closed ball:

**Theorem 2.11.** [2] Suppose that \((X, d)\) is a complete complex valued metric space and \( x_0 \in X \). Let \( 0 < r \in \mathbb{C} \) and \( A, B, C, D, E \) be five nonnegative real numbers such that \( A + B + C + 2D + 2E < 1 \). Let \( S, T : X \rightarrow X \) satisfy the inequality:

\[
d(Sx, Ty) \lambda \leq Ad(x, y) + B[d(x, Sx)d(y, Ty)]/[1 + d(x, y)] + C[d(y, Ty)d(x, Sx)]/[1 + d(x, y)] + D[d(x, Sx)d(y, Ty)]/[1 + d(x, y)] + E[d(y, Ty)d(x, Sx)]/[1 + d(x, y)],\]

\( \forall x, y \in B(x_0, r). \) \( \lambda \leq \max \{(A + D)/(1 - B - D), \, (A + E)/(1 - B - E)\} \).

If \( \lambda = \max \{(A + D)/(1 - B - D), \, (A + E)/(1 - B - E)\} \), then there exist a unique point \( u \in B(x_0, r) \) such that \( u = Su = Tu \).

**3. MAIN RESULTS**

**Theorem 3.1.** Let \( A, B, S, T: X \rightarrow X \) be four self-mappings of a complex valued b-metric space \((X, d)\) satisfying:

(i) \( A(X) \subseteq T(X) \), \( B(X) \subseteq S(X) \);

(ii) A set of rational inequalities:

\[
d(Ax, By) \leq q U_{xy}(A, B, S, T)...... \)

\( \forall x, y \in X \), \( q \) is a non-negative real number such that \( 0 \leq q < 1/(s^2 + s) \), with \( s \geq 1 \), and \( U_{xy}(A, B, S, T) \in \{d(Sx, Ty), d(Ax, Sx)d(By, Ty)/[d(Sx, Ty) + d(Ax, By)]\}

\( d(By, Sx)d(Ax, Ty)/[d(Sx, Ty) + d(Ax, By)], d(Ax, Sx)d(By, Sx)/[1 + d(Sx, Ty)]\),
(iii) Both the pairs (A, S) and (B, T) are weakly compatible.
(iv) Either the pair (B, T) satisfies property (E.A) and T(X) is complete, or the pair (A, S) satisfies the property (E.A) and T(X) is complete. Then mappings A, B, S and T have a unique common fixed point in X.

**Proof.** First suppose that the pair (B, T) satisfy property (E.A). Then, there exist a sequence \( \{x_n\} \) in X such that \( \lim_{n \to \infty} Bx_n = \lim_{n \to \infty} Tx_n = t \), for some \( t \in X \). Further, since \( B(X) \subseteq S(X) \), there exist a sequence \( \{y_n\} \) in X such that \( Bx_n = Sy_n \). So, \( \lim_{n \to \infty} Sy_n = t \). We claim that \( \lim_{n \to \infty} Ay_n = t \). If not, then putting \( x = y_n, y = x_n \) in condition (ii), we have

\[
d(Ay_n, Bx_n) \leq q \mu y_n, x_n (A, B, S, T) \ldots (3.2)
\]

where \( 0 \leq q < 1 \) / \( s^2 + s \), with \( s \geq 1 \), and

\[
\begin{align*}
Uy_n, x_n(A, B, S, T) & \in d(Sy_n, T_x), \\
d(Ay_n, Sy_n) & \in d(Bx_n, T_x)/[d(Sy_n, T_x) + d(Ay_n, Bx_n)], \\
d(Bx_n, Sy_n) & \in d(Ay_n, Bx_n)/[d(Sy_n, T_x) + d(Ay_n, Bx_n)]
\end{align*}
\]

We have following cases to consider:

**Case 1.** If \( d(Sy_n, T_x) \) is chosen in (3.2), then we have

\[
d(Ay_n, Bx_n) \leq q \mu d(Sy_n, T_x)
\]

\[
\Rightarrow |d(Ay_n, Bx_n)| \leq q |d(Sy_n, T_x)|.
\]

Letting \( n \to \infty \) it yields:

\[
\lim_{n \to \infty} |d(Ay_n, Bx_n)| = \lim_{n \to \infty} |d(Ay_n, t)| = 0 \Rightarrow \lim_{n \to \infty} Ay_n = t.
\]

**Case 2.** If option \( d(Ay_n, Sy_n), d(Bx_n, T_x) / [d(Sy_n, T_x) + d(Ay_n, Bx_n)] \) is chosen in (3.2), then we have

\[
d(Ay_n, Bx_n) \leq q \mu d(Ay_n, Sy_n)d(Bx_n, T_x)/[d(Sy_n, T_x) + d(Ay_n, Bx_n)]
\]

\[
\Rightarrow d(Ay_n, Bx_n) \leq q \mu d(Ay_n, Bx_n)d(Bx_n, T_x)/[d(Sy_n, T_x) + d(Ay_n, Bx_n)] \quad \text{(since Sy_n = Bx_n)}
\]

\[
\Rightarrow d(Bx_n, T_x) + d(Ay_n, Bx_n) \leq q \mu d(Ay_n, Bx_n)d(Bx_n, T_x)/[d(Sy_n, T_x) + d(Ay_n, Bx_n)]
\]

Letting \( n \to \infty \) it yields:

\[
\lim_{n \to \infty} |d(Ay_n, t)| \leq \lim_{n \to \infty} |d(Bx_n, T_x)| = 0 \Rightarrow \lim_{n \to \infty} Ay_n = t.
\]

**Case 3, 4, 7, 8.** If 0 is chosen in (3.2), then \( d(Ay_n, Bx_n) \leq 0 \Rightarrow |d(Ay_n, Bx_n)| \leq 0 \).

Letting \( n \to \infty \) it yields \( \lim_{n \to \infty} |d(Ay_n, t)| = 0 \Rightarrow \lim_{n \to \infty} Ay_n = t \).

**Case 5.** If \( d(Ay_n, T_x), d(Bx_n, T_x) / [1 + d(Sy_n, T_x)] \) is chosen in (3.2), then we have

\[
d(Ay_n, Bx_n) \leq q \mu d(Ay_n, T_x)d(Bx_n, T_x)/[1 + d(Sy_n, T_x)]
\]

\[
\leq q \mu [d(Ay_n, Bx_n)d(Bx_n, T_x)/[1 + d(Sy_n, T_x)]
\]

\[
\quad \text{(using (C3) and Sy_n = Bx_n)}
\]

\[
\Rightarrow |d(Ay_n, Bx_n)| \leq q \mu |d(Ay_n, Bx_n)d(Bx_n, T_x)| / [1 + d(Bx_n, T_x)]
\]

\[
\Rightarrow |d(Ay_n, Bx_n)| \leq q \mu |d(Bx_n, T_x)| / [1 + d(Bx_n, T_x)]
\]
\[ \begin{align*}
&< qs|d(Ay_n,Bx_n)+d(Bx_n,Tx_n)|, \\
&\quad (\because |d(Bx_n,Tx_n)| < |1+d(Bx_n,Tx_n)|). \\
&\Rightarrow |d(Ay_n,Bx_n)| < [qs/(1-qs)].|d(Bx_n,Tx_n)|. \\
\text{Letting } n \to \infty \text{ it yields:} \\
&\lim_{n \to \infty} |d(Ay_n,Bx_n)| \leq [qs/(1-qs)].0 \\
&\Rightarrow \lim_{n \to \infty} Ay_n = t.
\end{align*} \]

**Case 6.** If option \( d(Ay_n, Sy_n) \) \( d(Bx_n, Tx_n) \) / [1+d(Ay_n,Bx_n)] is chosen in (3.2), then, since \( Bx_n = Sy_n \), we have

\[ d(Ay_n,Bx_n) \leq q.d(Ay_n,Bx_n).d(Bx_n,Tx_n)/ [1+d(Ay_n,Bx_n)] \]

\[ \Rightarrow |d(Ay_n,Bx_n)| \leq q.|d(Bx_n,Tx_n)|. \]

\[ \{d(Ay_n,Bx_n)|/ [1+d(Ay_n,Bx_n)] \} \]

\[ < q.|d(Bx_n,Tx_n)| < |d(Bx_n,Tx_n)|; \]

\[ (\because |d(Ay_n,Bx_n)| < [1+d(Ay_n,Bx_n)]. \]

Letting \( n \to \infty \), and since \( \lim_{n \to \infty} |d(Bx_n,Tx_n)| = 0 \), it yields

\[ \lim_{n \to \infty} |d(Ay_n,Bx_n)| = \lim_{n \to \infty} |d(Ay_n, t)| \leq 0 \]

\[ \Rightarrow \lim_{n \to \infty} Ay_n = t. \]

**Case 9.** If \( d(Ay_n, Sy_n) \) \( d(Sy_n, Tx_n) \) / [1+d(Sy_n,Tx_n)] is chosen in (3.2), then we have

\[ d(Ay_n,Bx_n) \leq q.d(Ay_n,Bx_n).d(Sy_n,Tx_n)/ [1+d(Sy_n,Tx_n)] \]

\[ = q.d(Ay_n,Bx_n)d(Bx_n,Tx_n)/[1+d(Bx_n,Tx_n)], \]

(\( \because \) \( Sy_n = Bx_n \))

\[ \Rightarrow |d(Ay_n,Bx_n)| \leq q.|d(Ay_n,Bx_n)|.|d(Bx_n,Tx_n)|/ [1+d(Bx_n,Tx_n)] \]

\[ < q.|d(Ay_n,Bx_n)|; \]

\[ (\because |d(Bx_n,Tx_n)| < |1+d(Bx_n,Tx_n)|, \text{ and } q < 1). \]

Letting \( n \to \infty \), and since \( \lim_{n \to \infty} Bx_n = t \) it yields:

\[ \lim_{n \to \infty} |d(Ay_n, t)| \leq q.\lim_{n \to \infty} |d(Ay_n, t)| \]

\[ < \lim_{n \to \infty} |d(Ay_n, t)|, \]

a contradiction. Thus \( \lim_{n \to \infty} Ay_n = t. \)

**Case 10.** If \( d(Sy_n, Tx_n) \) \( d(Bx_n, Tx_n) \) / [d(Ay_n,Bx_n)+d(Sy_n,Tx_n)] is chosen in (3.2), then we have

\[ d(Ay_n,Bx_n) \leq q.d(Sy_n,Tx_n).d(Bx_n,Tx_n)/ [d(Ay_n,Bx_n)+d(Sy_n,Tx_n)] \]

\[ = q.d(Bx_n,Tx_n).d(Bx_n,Tx_n)/ [d(Ay_n,Bx_n)+d(Bx_n,Tx_n)] \]

\[ \Rightarrow \lim_{n \to \infty} Ay_n \leq q.|d(Bx_n,Tx_n)|.|d(Bx_n,Tx_n)|/ [d(Ay_n,Bx_n)+d(Bx_n,Tx_n)] \]

\[ \leq q.|d(Bx_n,Tx_n)|.|d(Bx_n,Tx_n)|/|d(Bx_n,Tx_n)|; \]

\[ (\because \) |d(Ay_n,Bx_n)| + d(Bx_n,Tx_n)| \geq |d(Bx_n,Tx_n)| \Rightarrow |d(Ay_n,Bx_n)| + d(Bx_n,Tx_n)|^{-1} \leq |d(Bx_n,Tx_n)|^{-1} \]

\[ \Rightarrow |d(Ay_n,Bx_n)| \leq q.|d(Bx_n,Tx_n)| < |d(Bx_n,Tx_n)|. \]

Letting \( n \to \infty \), and since \( \lim_{n \to \infty} |d(Bx_n,Tx_n)| = 0 \) it yields:

\[ \lim_{n \to \infty} |d(Ay_n,Bx_n)| = \lim_{n \to \infty} |d(Ay_n, t)| \leq 0 \]

\[ \Rightarrow \lim_{n \to \infty} Ay_n = t. \]

Therefore, in all cases, we obtain \( \lim_{n \to \infty} Ay_n = t. \) Hence

\[ \lim_{n \to \infty} Ay_n = \lim_{n \to \infty} Sy_n = \lim_{n \to \infty} Bx_n = \lim_{n \to \infty} Tx_n = t..........................(3.3) \]

Now, suppose first that \( S(X) \) is a complete subspace of \( X \), then \( t = Su, \) for some \( u \in X. \) We claim that \( Au = t. \) If not, then putting \( x = u, y = x_n \) in (ii), we have

\[ d(Au,Bx_n) \leq q.U_{x_n}(A,B,S,T)......(3.4) \]

where \( U_{x_n}(A, B, S, T) \in \{d(t, Tx_n), d(Au,t).d(Bx_n,Tx_n)/[d(t,Tx_n)+d(Au,Bx_n), \]

\[ d(Bx_n,t).d(Au,Tx_n)/[d(t,Tx_n)+d(Au,Bx_n), \]

\[ d(Au,t).d(Bx_n,t)/[1+d(t,Tx_n)], \]
\[d(Au, Tx_n) d(Bx_n, Tx_n) \big/ \[1 + d(t, Tx_n)\],
\]
\[d(Au, t) d(Bx_n, Tx_n) \big/ \[1 + d(Au, Bx_n)\],
\]
\[d(t, Bx_n) d(Au, Bx_n) \big/ \[1 + d(Au, Bx_n)\],
\]
\[d(Bx_n, t) d(t, Tx_n) \big/ \[1 + d(t, Tx_n)\],
\]
\[d(Au, t) d(t, Tx_n) \big/ \[1 + d(t, Tx_n)\],
\]
\[d(t, Bx_n) d(Bx_n, Tx_n) \big/ [1 + d(t, Tx_n)],
\]
\[d(t, Tx_n) d(Bx_n, Tx_n) \big/ [1 + d(Au, Bx_n) + d(t, Tx_n)].
\]

If \( n \to \infty \) then each co-ordinate within \( \cup_{u, x, n} (A, B, S, T) \) tends to zero, as in above

Cases 1-10. So, that \( \lim_{n \to \infty} |d(Au, Bx_n)| \leq q(0,0,0,0,0,0,0,0,0,0)=0 \).

Thus, in all cases \( \lim_{n \to \infty} |d(Au, Bx_n)| = \lim_{n \to \infty} |d(Au, t)| \leq 0 \Rightarrow Au = t. \)

Therefore \( Au = Su = t, \) and \( u \) is a coincidence point of \( (A, S) \). Further, since \( (A, S) \) is weakly compatible so that \( ASu = SAu, i.e. At = St. \)

On the other hand, since \( A(X) \subseteq T(X) \), there exist \( v \) in \( X \) such that \( Au = Tv. \)

Thus \( Au = Su = Tv = t. \) We claim that \( Bv = t. \) If not, then putting \( x = u, y = v \)
in condition (ii), we have

\[d(t, Bv) = d(Au, Bv) \leq q(U_{u,v}(A, B, S, T)) \ldots (3.5)\]

Where

\[U_{u,v}(A, B, S, T) \in \{d(Su, Tv), d(Au, Su) d(Bv, Tv) \big/ [d(Su, Tv) + d(Au, Bv)],
\]
\[d(Bv, Su) d(Au, Tv) \big/ [d(Su, Tv) + d(Au, Bv)],
\]
\[d(Au, Su) d(Bv, Su) \big/ [1 + d(Su, Tv)],
\]
\[d(Au, Tv) d(Bv, Tv) \big/ [1 + d(Su, Tv)],
\]
\[d(Au, Su) d(Bv, Tv) \big/ [1 + d(Au, Bv)],
\]
\[d(Su, Bv) d(Au, Bv) \big/ [1 + d(Au, Bv)],
\]
\[d(Bv, Su) d(Su, Tv) \big/ [1 + d(Su, Tv)],
\]
\[d(Au, Su) d(Su, Tv) \big/ [1 + d(Su, Tv)],
\]
\[d(Su, Tv) d(Bv, Tv) \big/ [d(Au, Bv) + d(Su, Tv)].
\]

\[=\{0,0,0,0,0,0,0,0\}=0\]

Thus from (3.5), we have \( d(t, Bv) \leq q.0 \leq |d(t, Bv)| = 0 \Rightarrow Bv = t. \) Thus \( Bv = Tv = t, \) and \( v \) is a coincidence point of \( (B, T). \) Further, since \( (B, T) \)
is weakly compatible, so that \( BTv = TBv, \) or, \( Bt = Tt. \) Thus \( At = St, Bt = Tt. \)

Next, we show that \( At = Bt. \) For, putting \( x = t, y = t \)
in (ii), we have

\[d(At, Bt) \leq qU_{u,t} (A, B, S, T) \ldots (3.6)\]

Where

\[U_{u,t}(A, B, S, T) \in \{d(St, Tt), d(At, St) d(Bt, Tt) \big/ [d(St, Tt) + d(At, Bt)],
\]
\[d(Bt, St) d(At, Tt) \big/ [d(St, Tt) + d(At, Bt)],
\]
\[d(At, St) d(Bt, St) \big/ [1 + d(St, Tt)],
\]
\[d(At, Tt) d(Bt, Tt) \big/ [1 + d(St, Tt)],
\]
\[d(At, St) d(Bt, Tt) \big/ [1 + d(At, Bt)],
\]
\[d(St, Bt) d(At, Bt) \big/ [1 + d(At, Bt)],
\]
\[d(St, Tt) d(St, Tt) \big/ [1 + d(St, Tt)],
\]
\[d(St, Tt) d(Bt, Tt) \big/ [d(At, Bt) + d(St, Tt)];
\]
\[=\{d(At, Bt), 0, ½d(At, Bt), 0, 0, 0,
\]
\[d(At, Bt) d(At, Bt) \big/ [1 + d(At, Bt)],
\]
\[d(At, Bt) d(At, Bt) \big/ [1 + d(At, Bt)],
\]

.\)

We have following cases to consider:

Case 1. If we chose first co-ordinate \( d(At, Bt) \),
then inequality (3.6) reduces to:

\[d(At, Bt) \leq qd(At, Bt)
\]
\[\Rightarrow |d(At, Bt)| \leq q|d(At, Bt)| < |d(At, Bt)|,
\]
Which is a contradiction. Thus \( At = Bt. \)

Case 2,4,5,6,9,10. If we chose 0, then inequality (3.6) reduces to:

\[d(At, Bt) \leq q.0 \Rightarrow |d(At, Bt)| \leq 0 \Rightarrow At = Bt.
\]

Case 3. If we choose third co-ordinate
\[½.d(At, Bt), \]
then inequality (3.6) reduces to:
\[ d(At, Bt) \leq q \cdot \frac{1}{2} d(At, Bt) \]
\[ \Rightarrow |d(At, Bt)| \leq \frac{1}{2} q |d(At, Bt)| < |d(At, Bt)|, \]
a contradiction. Thus \( At = Bt. \)

**Case 7, 8.** If we chose option
\[ d(At, Bt) \cdot d(At, Bt)/[1 + d(At, Bt)] \]
then the inequality (3.6) reduces to:
\[ d(At, Bt) \leq q.d(At, Bt).d(At, Bt)/[1 + d(At, Bt)] \]
\[ \Rightarrow |d(At, Bt)| \leq q.|d(At, Bt)|/\left\{ d(At, Bt)/[1 + d(At, Bt)] \right\} < |d(At, Bt)|, \]
which is a contradiction (since \(|d(At, Bt)| < 1 + d(At, Bt)|, and \( q < 1 \)). Thus \( At = Bt. \)

Hence in all cases \( At = Bt. \) Thus \( At = Bt = St = Tt, \) i.e., \( t \) is a coincidence point of \( A, B, S, T. \) Similar cases arises if we consider that the pair \( (A, S) \) satisfy property \((E.A)\) and the set \( T(X) \) is complete. So, in both cases \( At = Bt = St = Tt. \)

Now, we claim that \( t \) is a common fixed point of \( A, B, S, T. \) For, putting \( x = t, y = v \) in (ii), and using \( Bv = Tt, \) where \( Bv = t, \) we have
\[ d(At, t)=d(At, Bv)\leq q U_{t,v}(A, B, S, T) \]...

where \( U_{t,v}(A, B, S, T) \in \{ d(St, Tt), d(At, St), d(Bv, Tt) \}, \)
\[ d(At, Bv)/[1 + d(At, Bv)], \]
\[ d(Bv, St)/[1 + d(Bv, St)], \]
\[ d(At, Tt)/[1 + d(At, Tt)], \]
\[ d(At, Bv)/[1 + d(At, Bv)], \]
\[ d(Bv, Tt)/[1 + d(Bv, Tt)], \]
\[ d(At, Tt)/[1 + d(At, Tt)], \]
\[ d(At, Bv)/[1 + d(At, Bv)], \]
\[ d(At, Tt)/[1 + d(At, Tt)], \]
\[ d(At, Bv)/[1 + d(At, Bv)], \]
\[ u_{a, t}(A, B, S, T) \]
\[ = \{d(At, t), 0, \frac{1}{2} d(At, t), 0, \}
\[ d(At, t)/[1 + d(At, t)], 0, \]
\[ d(At, t)/[1 + d(At, t)], 0, 0 \}
\[ . \]
We have following cases to consider:

**Case 1.** If we choose first co-ordinate \( d(At, t), \) then inequality (3.7) reduces to:
\[ d(At, t) \leq q d(At, t) \]
\[ \Rightarrow |d(At, t)| \leq q |d(At, t)| < |d(At, t)|, \]
which is a contradiction. Thus \( At = t. \)

**Case 2, 4, 6, 9, 10.** If we choose 0, then the inequality (3.7) reduces to:
\[ d(At, t) \leq q 0 \]
\[ \Rightarrow |d(At, t)| \leq 0 \]
\[ \Rightarrow At = t. \]

**Case 3.** If we chose third co-ordinate \( \frac{1}{2} d(At, t), \) then inequality (3.7) reduces to:
\[ d(At, t) \leq q \cdot \frac{1}{2} d(At, t) \]
\[ \Rightarrow |d(At, t)| \leq \frac{1}{2} q |d(At, t)| < |d(At, t)|, \]
a contradiction (since \( |d(At, t)| < 1 + d(At, t)\), and \( q < 1 \)). Thus \( At = t. \)

**Case 5, 7, 8.** If we chose
\[ d(At, t).d(At, t)/[1 + d(At, t)] \]
then inequality (3.7) reduces to:
\[ d(At, t) \leq q \cdot \frac{1}{2} d(At, t) \]
\[ \Rightarrow |d(At, t)| \leq \frac{1}{2} q |d(At, t)| < |d(At, t)|, \]
a contradiction (since \( |d(At, t)| < 1 + d(At, t)\), \( q < 1 \)). Thus \( At = t. \)

Lastly, we show that \( t \) is the unique common fixed point of \( A, B, S, T. \) If, \( w \neq t \) be another common fixed point, then putting \( x = t, y = w \) in (ii), we have
\[ d(t, w)=d(At, Bw)\leq q U_{t,w}(A, B, S, T) \]...

where \( q \) is a non-negative real number such that
\[ 0 \leq q < 1/(s^2 + s) \] with \( s \geq 1 \) and
\[ U_{t,w}(A, B, S, T) \in \{ d(St, Tt), \}
\[ d(At, t).d(At, Tt)/[d(St, Tt) + d(At, Bw)], \]
Hence in all cases \( t = w \). This shows the uniqueness of common fixed point \( t \) for four mappings \( A, B, S, T \) in the set \( X \). This completes the proof.

If \( A = B = f \) and \( S = T = g \) in Theorem 3.1, we have the following result:

**Corollary 3.2.** Let \( f, g : X \to X \) be two self-mappings of a complex valued \( b \)-metric space \((X, d)\) such that:

(i) \( f(X) \subseteq g(X) \), where \( g(X) \) is complete;  
(ii) \( d(fx, fy) \leq q \cup_{x,y}(f, g) \), \( \forall x, y \in X \)...

Where \( q \) is a non-negative real number such that \( 0 \leq q < 1/\left(s^2+s\right) \), \( s \geq 1 \), and

\(
\cup_{x,y}(f, g) = \{d(gx, gy), d(fx,gx) \times d(fy,gy) / \left[d(gx,gy)+d(fx,fy)\right],
\}
\)

If the pair \( (f, g) \) is weakly compatible and satisfy property (E.A), then mappings \( f, g \) have a unique common fixed point in \( X \).

**DISCUSSION** If \( A = f, B = g \) and \( S = T = I \), the identity mapping, then the condition of completeness of \( S(X) \), or \( T(X) \) becomes the completeness of whole space \( X \). The set-inclusion condition \( A(X) \subseteq T(X) \) and \( B(X) \subseteq S(X) \) reduces to \( f(X) \subseteq X \) and \( g(X) \subseteq X \), and this obvious condition can be omitted.

Further, the (E.A) property of pair \( (A, S) \) in Theorem 3.1 reduces to the following conditions (1) and (2):

(1)“There exist a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} fx_n = \lim_{n \to \infty} x_n = t = x_0 \) (say), for some \( t \in X \)."
(2) Further, \( \lim_{n \to \infty} f x_n = x_0 \) can be expressed as
\[
\text{"there exist a point } x_0 \in X \text{ with } 0 < r \in C \text{ such that } |d(x_0, f x_n)| \leq |r|".
\]

Unifying (1) and (2) we get condition (3): “A closed ball centered at \( x_0 \), namely
\[
B(x_0, r)
\]
, is assumed; where the mapping \( f \) satisfy the condition \( |d(x_0, f x_n)| \leq |r| \), with \( f x_n \to x_0 \) where \( \{x_n\} \) is a sequence converging to \( x_0 \in X \), i.e., condition (3):
\[
|d(x_0, f x_n)| \leq |r|, \text{ for all } x_n \in B(x_0, r)
\]

Similarly, we can write for the (E.A) property of pair \( (B, T) \).

On the basis of condition (1), Theorem 3.1 can be written as the following result:

**Corollary 3.3.** Let \( f, g : X \to X \) be two self-maps of a complete complex valued \( b \)-metric space \( (X, d) \) satisfying a set of rational inequalities
\[
d(f(x), g(y)) \leq q \cup_{x \in X} (f, g), \forall x, y \in X \ldots \quad (3.10)
\]

where \( q \) is a non-negative real number such that \( 0 \leq q < 1/(s^2 + s) \), \( s \geq 1 \) and
\[
\cup_{x \in X} (f, g) \in \{d(x, y), d(f(x), g(y))/[d(x, y) + d(f(x), g(y))], d(g(x), f(y))/[d(x, y) + d(f(x), g(y))], d(f(x), g(y))/[1 + d(f(x), g(y))], d(g(x), f(y))/[1 + d(f(x), g(y))]\}
\]

If there exist a sequence \( \{x_n\} \) in \( X \) such that \( \lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = t \), or \( \lim_{n \to \infty} g x_n = \lim_{n \to \infty} f x_n = t \) for some \( t \in X \); then mappings \( f, g \) have a unique common fixed point in \( X \).

If we restrict ourselves to the contractiveness of the mappings \( f, g \) only on a closed ball, as in Ahmad et al. [2], instead of the whole space; then on the basis of condition (3), our Theorem 3.1 can be written as the following result:

**Corollary 3.4.** Suppose that \( (X, d) \) is a complete complex valued \( b \)-metric space and \( x_0 \in X \). Let \( 0 < r \in C \). Let \( f, g : X \to X \) satisfy a set of rational inequalities:
\[
d(f(x), g(y)) \leq q \cup_{x \in X} (f, g), \forall x, y \in B(x_0, r) \ldots (3.11)
\]

where \( q \) is a non-negative real number such that \( 0 \leq q < 1/(s^2 + s) \), \( s \geq 1 \) and
\[
\cup_{x \in X} (f, g) \in \{d(x, y), d(f(x), g(y))/[d(x, y) + d(f(x), g(y))], d(g(x), f(y))/[d(x, y) + d(g(x), g(y))]\}
\]

If \( |d(x_0, f x_0)| \leq kr \), \( k < 1 \), then there exist a unique point \( z \in B(x_0, r) \) such that \( z = f z = g z \).

If we put \( A = S, B = T, S = T = I \), the identity mapping, then set-inclusion condition (i) of Theorem 3.1 reduces to:
\[
S(X) \subseteq X, T(X) \subseteq X, \text{ which is always true for any mapping.}
\]
Similarly, the weakly compatibility condition (ii) reduces to:
\[
S I x = I z \text{ whenever } S x = I x, \text{ and } T I x = T I x
\]
whenever \( T x = I x = z \), which are obvious for any mapping. So, we can drop conditions (i), (iii).
Also, completeness of range subspaces \( S(X) \) or \( T(X) \) of condition (iv) reduces to completeness of whole space \( X \). Thus our Theorem 3.1 reduces to the following result:

**Corollary 3.5.** Let \( S, T : X \to X \) be two self-maps of a complete complex valued \( b \)-metric space \( (X, d) \) satisfying:
\[
d(S x, T y) \leq q \cup_{x \in X} (S, T), \forall x, y \in X \ldots (3.12)
\]
where q is a non-negative real number such that $0 \leq q < 1/(s^2+s)$, $s \geq 1$ and

\[
\begin{align*}
U_{xy}(f, g) &\in \{d(x, y), \\
d(Sx, x) d(Ty, y) / [d(x, y) + d(Sx, Ty)], \\
d(Sx, x) d(Ty, x) / [1 + d(x, y)], \\
d(Sx, y) d(Ty, y) / [1 + d(Sx, Ty)], \\
d(Ty, y) d(x, y) / [1 + d(x, y)], \\
d(Sx, x) d(Ty, x) / [1 + d(Sx, Ty)], \\
d(Sx, x) d(Ty, x) / [1 + d(x, y)], \\
d(Sx, x) d(Ty, x) / [1 + d(Sx, Ty)], \\
d(Ty, x) d(x, y) / [1 + d(x, y)], \\
d(Sx, y) d(x, y) / [1 + d(x, y)], \\
d(Sx, x) d(Ty, x) / [1 + d(Sx, Ty)], \\
d(x, Ty) d(Sx, Ty) / [1 + d(Sx, Ty)], \\
d(y, Ty) d(x, y) / [1 + d(x, y)], \\
d(Sx, x) d(Ty, x) / [1 + d(Sx, Ty)], \\
d(Sx, x) d(Ty, x) / [1 + d(x, y)], \\
d(Ty, x) d(x, y) / [1 + d(x, y)], \\
d(Sx, x) d(x, y) / [1 + d(x, y)], \\
d(Sx, x) d(Ty, x) / [1 + d(Sx, Ty)], \\
d(x, Ty) d(Sx, Ty) / [1 + d(Sx, Ty)], \\
d(y, Ty) d(x, y) / [1 + d(x, y)].
\end{align*}
\]

If there be a sequence $\{x_n\}$ in $X$ such that $\lim_{n \to \infty} Tx_n = \lim_{n \to \infty} x_n = t$ for some $t \in X$ then mappings $S$ and $T$ have a unique common fixed point in $X$.

REFERENCES


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